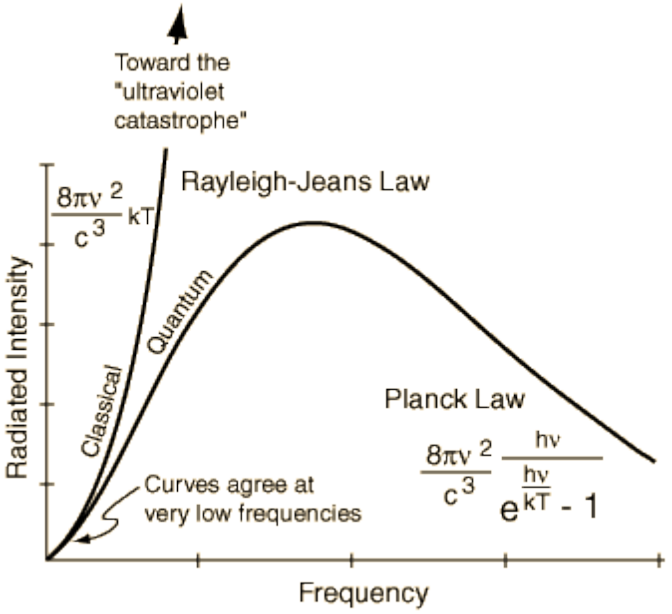
The Photon Gas

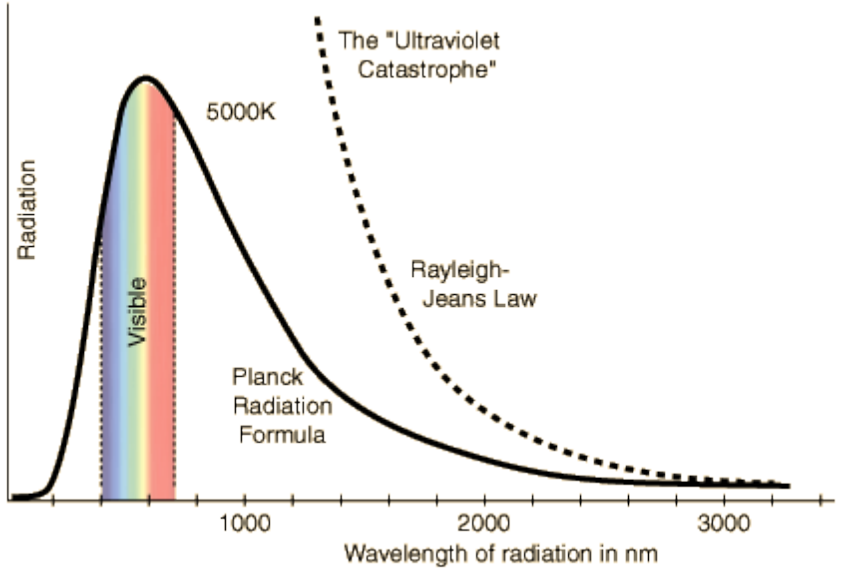
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Planck's Function





http://hyperphysics.phy-astr.gsu.edu/hbase/mod6.html#c4

http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

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Photon Gas

- Photons can be treated as an ideal gas with two exceptions:
  - Photons are bosons and Bose-Einstein statistics must be used. However, photons do not interact with each other so that no approximation is made by neglecting interparticle forces.
  - Photons can be absorbed and emitted by the walls of the container, so that no constraint can be placed on the number of photons even in the Grand Canonical representation. ($\gamma = 0$)
- We will derive the energy distribution, i.e. Planck's function.

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Recall that for a non-interacting gas

$$N_k = \frac{g_k}{e^{\gamma + \varepsilon_k / kT} - 1}$$

However, since the constraint on N is removed, this becomes

$$N_k = \frac{g_k}{e^{\varepsilon_k / kT} - 1}$$

If we make the continuous approximation, then

$$dN = \frac{dg}{e^{\varepsilon / kT} - 1}$$

where dg is the degeneracy for the energy increment ε to $\varepsilon + d\varepsilon$.

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The degeneracy is almost the same as for a particle in a box:

$$dg = 2\left(\frac{4\pi n^2 dn}{8}\right)$$

where the factor of two arises because an electromagnetic field can be polarized in two independent directions.

For a molecular gas, we related dn to $d\varepsilon$ using Newtonian physics, namely that

$$\varepsilon = \frac{mc^2}{2} \qquad \text{or} \qquad p^2 = 2m\varepsilon$$

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However, photons are relativistic, and de Broglie's relation

$$p = \frac{\varepsilon}{c}$$

must be used. Therefore,

$$\varepsilon^2 = \frac{h^2 c^2}{4V^{2/3}} n^2$$

so

$$dg = \frac{8\pi V}{h^3 c^3} \varepsilon^2 d\varepsilon$$

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Subbing this into the expression for dN and noting that

$$\varepsilon = h\nu$$

We get for the number distribution

$$\frac{dN}{V} = \frac{8\pi}{c^3}\frac{\nu^2}{(e^{h\nu/kT} - 1)}d\nu$$

This is the number of photons (per unit volume) in the frequency range ν to $\nu + d\nu$.

However, we seek the spectral energy density

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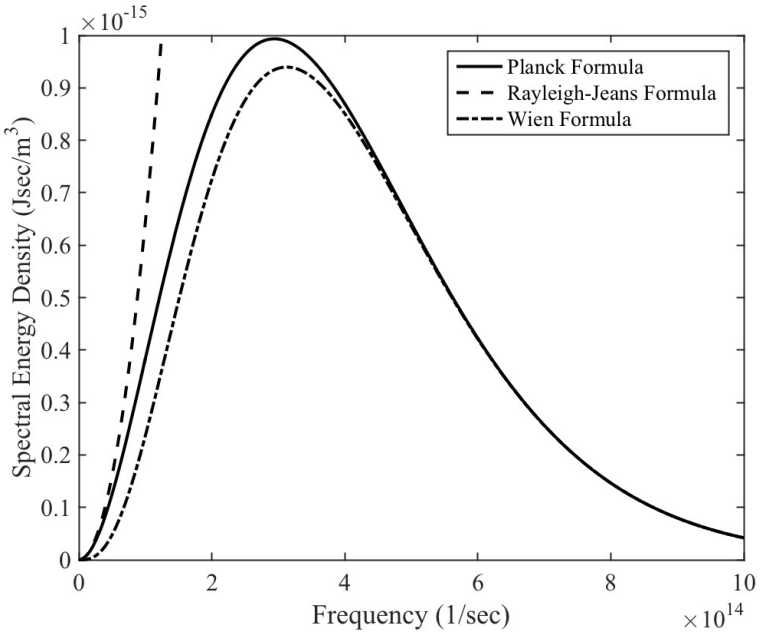
Since

$$u_\nu d\nu = h\nu \frac{dN}{V}$$

then

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{(e^{-h\nu/kT} - 1)}$$

This is Planck's Law



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The limits for low and high frequency are readily obtained. For large ν, and $h\nu / kT >> 1$ we get the Wien Formula:

$$u_\nu \cong \frac{8\pi h\nu^3}{c^3} e^{-h\nu/kT}$$

For small ν, and $h\nu / kT << 1$ we get the Rayleigh-Jeans Formula:

$$u_\nu \cong \frac{8\pi\nu^2 kT}{c^3}$$

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The total energy per unit volume is

$$u = \int_0^\infty u_\nu d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{(e^{-h\nu/kT} - 1)} d\nu = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}$$

We could have obtained this relation by first evaluating the partition function, thereby obtaining the fundamental relation. Energy is then one of the equations of state in the Grand Canonical Representation. For a boson and with $\gamma = 0$

$$\ln Q = \sum_k \ln \frac{1}{(1 - e^{-\beta\varepsilon_k})} \cong \int_0^\infty g(\nu) \ln \frac{1}{(1 - e^{-h\nu/kT})} d\nu = \frac{8\pi^5 V}{45(hc/kT)^3}$$

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Therefore, the Fundamental Relation becomes

$$S[1/T,0] = \frac{8k\pi^5 V}{45(hc/kT)^3} = \frac{PV}{T}$$

The energy we have already derived. The pressure is

$$p = \frac{\partial S[1/T]}{\partial V}\Big)_{1/T} = \frac{8\pi^5}{45(hc)^3}(kT)^4$$

To obtain the number of photons requires differentiating with respect to μ before setting $\mu = 0$

$$N = -\frac{\partial S[1/T,-\mu/T]}{\partial \mu/T}\Big)_{1/T,V} = 1.202\frac{16\pi V}{(hc/kT)^3}$$

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The entropy for the photon gas is

$$S = \frac{32k\pi^5 V}{45(hc/kT)^3}$$

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Finally, we can also derive the important Stefan-Boltzmann relation used in heat transfer. For an ideal gas, the flux of particles across any given plane is $n\bar{c}/4$. This relation is derived by integration of the Maxwellian velocity distribution function. By analogy, we can show that the energy flux of photons (energy per unit time and area) is

$$q = \frac{2\pi^5 k^4}{15h^3c^2}T^4 = \sigma T^4$$

Where the Stefan-Boltzman constant is

$$\sigma = 2\frac{\pi^5}{15}\frac{k^4}{h^3c^2}T^4 = 5.669 \times 10^{-8}\mathrm{W/m^2K^4}$$

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